Correction Exam Calcul matriciel - Avril 2024

Ex 1 (4 pts)

1) $\det\begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix} = 1\times 2 - (2\times(-1)) = 4 \neq 0$ donc A est inversible.

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0,75 $\det\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} = 1\times(-2) - 2\times(-1) = 0$ donc B non inversible.

2) $\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ 2 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)+2 = \lambda^2 - 3\lambda + 4$

$\Delta = 9 - 16 < 0$ pas de racines pas de vp.

1,25 A non diagonalisable.

$\det(B-\lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ 2 & -2-\lambda \end{vmatrix} = (1-\lambda)(-2-\lambda)+2 = \lambda^2+\lambda = \lambda(\lambda+1)$

$\lambda = 0$ et $\lambda = -1$ sont vp.

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B a 2 vp $\neq$ donc B diagonalisable.

Ex 2 (7,5)

1) a) $\det A = \begin{vmatrix} m & 1 & 1 \\ 1 & m & m \\ 1 & 1 & m \end{vmatrix} \underset{L_2-L_3}{=} \begin{vmatrix} m & 1 & 1 \\ \boxed{0 \quad m-1 \quad 0} \\ 1 & 1 & m \end{vmatrix} = (m-1)\begin{vmatrix} m & 1 \\ 1 & m \end{vmatrix} = (m-1)(m^2-1)$

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$\det A = (m-1)(m-1)(m+1) = (m-1)^2(m+1)$

0,5 b) A inversible $\iff \det A \neq 0 \iff m \neq 1$ et $m \neq -1$.

c) Si A inversible, $r(A) = 3$

Si $m = 1$ $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ donc $r(A) = 1$

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Si $m = -1$ $r(A) = r\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{pmatrix} \underset{\substack{L_2+L_1 \\ L_3+L_1}}{=} r\begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} = r\begin{pmatrix} -1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 2$

2) (a) (S) est de Cramer $\iff m \neq 1$ et $m \neq -1$.

b) si $m = 0$, (S) est de Cramer. $\det A = 1$

1,5 $x = \dfrac{\begin{vmatrix} 1 & 1 & \boxed{1} \\ 1 & 0 & \boxed{0} \\ 0 & 1 & \boxed{0} \end{vmatrix}}{1} = 1$ $\quad y = \dfrac{\begin{vmatrix} 0 & 1 & \boxed{1} \\ 1 & 1 & \boxed{0} \\ 1 & 0 & \boxed{0} \end{vmatrix}}{1} = -1$ $\quad z = \dfrac{\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}}{1} \underset{C_2-C_3}{=} \dfrac{\begin{vmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{vmatrix}}{1} = 2$

Solution $(1, 1, 2)$

c) Cas $m = 1$ (S) $\iff x+y+z = 1$
$\iff z = -x-y$

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Donc les solutions sont du type $(x, y, -x-y)$, $x, y \in \mathbb{R}$.
Une inté de solutions

d) cas $m=-1$ $\quad (S) \Leftrightarrow \begin{cases} -x+y+z = 1 \\ x-y-z = 1 \\ x+y-z = -1 \end{cases} \quad \begin{cases} -x+z = 2 \\ x-z = 0 \\ L_3-L_2 : 2y = -2 \quad y=-1 \end{cases}$

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$\Leftrightarrow \begin{cases} x-z = -2 \\ x-z = 0 \\ y = -1 \end{cases}$ impossible donc pas de solution.

(2)

EX 3 1,5

1) $\det(A-\lambda I_3) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & -1-\lambda & 1 \\ -1 & 1 & -1-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & -1-\lambda & 1 \\ 0 & -\lambda & -\lambda \end{vmatrix}_{L_3+L_2}$

$= \begin{vmatrix} 1-\lambda & 0 & 1 \\ 1 & -2-\lambda & 1 \\ \boxed{0 \quad 0} & & -\lambda \end{vmatrix}_{C_2-C_3} = -\lambda(1-\lambda)(-2-\lambda)$

$= \lambda(1-\lambda)(2+\lambda)$

2) A admet 3 vps distincts en dim 3 donc diagonalisable.

0,5 $\begin{cases} \lambda_1 = 0 \\ \lambda_2 = 1 \\ \lambda_3 = -2 \end{cases}$

1,5 on résout $AX=0 \Leftrightarrow \begin{cases} x+y+z = 0 \\ x-y+z = 0 \\ -x+y-z = 0 \end{cases} \quad \begin{cases} L_1-L_2 : 2y = 0 \\ x = -z \end{cases} \quad E_0 = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} x,\ x \in \mathbb{R} \right\}$

$v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

on résout $AX = 1X \Leftrightarrow \begin{cases} x+y+z = x \\ x-y+z = y \\ -x+y-z = z \end{cases} \quad \begin{cases} z = -y \\ x = 3y \end{cases} \quad E_1 = \left\{ \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} y,\ y \in \mathbb{R} \right\} \quad v_2 = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$

on résout $AX = -2X \Leftrightarrow \begin{cases} x+y+z = -2x \\ x-y+z = -2y \\ -x+y-z = -2z \end{cases} \Leftrightarrow \begin{cases} 3x+y+z = 0 \\ x+y+z = 0 \\ -x+y+z = 0 \end{cases} \Leftrightarrow \begin{cases} L_2-L_3 : 2x = 0 \\ z = -y \end{cases}$

$E_{-2} = \left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} y,\ y \in \mathbb{R} \right\} \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

3) $P = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix}$

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4) Calculons $P^{-1} = \frac{1}{\det P} {}^t(\mathrm{com}\, P)$

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$\det P = \begin{vmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & -1 \end{vmatrix}_{L_3+L_1} = -3 \qquad \mathrm{com}(P) = \begin{pmatrix} +0 & -1 & +1 \\ -(3) & +(-1) & -2 \\ +3 & -1 & +1 \end{pmatrix} \qquad P^{-1} = \frac{1}{-3} \begin{pmatrix} 0 & 3 & 3 \\ -1 & -1 & -1 \\ 1 & -2 & 1 \end{pmatrix}$

$$P^{-1} = \begin{pmatrix} 0 & -1 & -1 \\ 1/3 & 1/3 & 1/3 \\ -1/3 & 2/3 & -1/3 \end{pmatrix}$$

5) $A = PDP^{-1}$ donc $A^n = PD^nP^{-1}$ (par rec sur n)

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$$A^n = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (-2)^n \end{pmatrix}\begin{pmatrix} 0 & -1 & -1 \\ 1/3 & 1/3 & 1/3 \\ -1/3 & 2/3 & -1/3 \end{pmatrix}$$

$$A^n = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 1 & (-2)^n \\ 0 & -1 & -(-2)^n \end{pmatrix}\begin{pmatrix} 0 & -1 & -1 \\ 1/3 & 1/3 & 1/3 \\ -1/3 & 2/3 & -1/3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1/3-\frac{1}{3}(-2)^n & 1/3+\frac{2}{3}(-2)^n & 1/3-\frac{1}{3}(-2)^n \\ -1/3+\frac{(-2)^n}{3} & -\frac{1}{3}-\frac{2}{3}(-2)^n & -\frac{1}{3}+\frac{1}{3}(-2)^n \end{pmatrix}$$

6) le système s'écrit $\underbrace{\begin{pmatrix} u_{n+1} \\ V_{n+1} \\ W_{n+1} \end{pmatrix}}_{X_{n+1}} = A \underbrace{\begin{pmatrix} u_n \\ V_n \\ W_n \end{pmatrix}}_{X_n}$ donc (rec sur n) $X_n = A^nX_0 = A^n\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

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$$u_n = 1+1+1 = 3$$
$$V_n = \frac{1}{3}+\frac{1}{3}+\frac{1}{3} = 1$$
$$W_n = \left(-\frac{1}{3}+\frac{1}{3}(-2)^n\right)+\left(-\frac{1}{3}-\frac{2}{3}(-2)^n\right)+\left(-\frac{1}{3}+\frac{1}{3}(-2)^n\right) = -1$$