

Séance 7

* Exercice 15. Soit f définie par $f(x,y) = \ln(1+x^2y^4)$.

Déterminer le développement limité à l'ordre 2 de f au point $A = (1; 1)$.

En déduire une valeur approchée de $f(B) - f(A)$ avec $B = (0.99; 1.02)$.

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$(x,y) \mapsto \ln(1+x^2y^4)$$

$$\forall (x,y) \in \mathbb{R}^2, \quad x^2y^4 \geq 0$$

$$\Rightarrow 1+x^2y^4 \geq 1 > 0$$

$$\text{et } \mathcal{D}_\ln =]0; +\infty[\Rightarrow$$

$$\mathcal{D} = \mathbb{R}^2.$$

f est la composée de fonctions $\mathcal{C}^2$ ($\ln$ et un polynôme)

Donc f est $\mathcal{C}^2$ sur $\mathcal{D} = \mathbb{R}^2$

$$(\ln u)' = \frac{u'}{u}$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} (\ln(1+x^2y^4)) = 2xy^4 \frac{1}{1+x^2y^4} = \frac{2xy^4}{1+x^2y^4}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (\ln(1+x^2y^4)) = 4x^2y^3 \frac{1}{1+x^2y^4} = \frac{4x^2y^3}{1+x^2y^4}$$

$$\frac{u'}{u} = \frac{u'v - uv'}{v^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{2xy^4}{1+x^2y^4} \right) = \frac{2y^4(1+x^2y^4) - 2xy^4 \cdot 2xy^4}{(1+x^2y^4)^2}$$

$$= \frac{2y^4[1+x^2y^4 - 2x^2y^4]}{(1+x^2y^4)^2} = \frac{2y^4(1-x^2y^4)}{(1+x^2y^4)^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y} \left(\frac{4x^2y^3}{1+x^2y^4} \right) = \frac{8x^2y^3(1+x^2y^4) - 4x^2y^3 \cdot 4x^2y^3}{(1+x^2y^4)^2}$$

$$= \frac{8x^2y^3(1+x^2y^4 - x^2y^4)}{(1+x^2y^4)^2} = \frac{8x^2y^3}{(1+x^2y^4)^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial y} \left(\frac{2xy^4}{1+x^2y^4} \right) = \frac{12x^2y^2(1+x^2y^4) - 4x^2y^2 \cdot 4x^2y^3}{(1+x^2y^4)^2}$$

$$= \frac{4x^2y^2(3+3x^2y^4 - 4x^2y^2)}{(1+x^2y^4)^2} = \frac{4x^2y^2(3-x^2y^2)}{(1+x^2y^4)^2}$$

$DL_2 f(A)$

$$f(A+H) = f(A) + \frac{\partial f}{\partial x}(A)h + \frac{\partial f}{\partial y}(A)k + \frac{1}{2} \left(\frac{\partial^2 f}{\partial x^2}(A)h^2 + 2 \frac{\partial^2 f}{\partial x \partial y}(A)hk + \frac{\partial^2 f}{\partial y^2}(A)k^2 \right) + \|H\|^2 \varepsilon(H)$$

$$\lim_{H \rightarrow 0} \frac{\varepsilon(H)}{\|H\|^2} = 0$$

en $A(1,1)$, $1+x^2y^4=1+1=2$

• $f(A) = \ln(2)$ ← intuitive

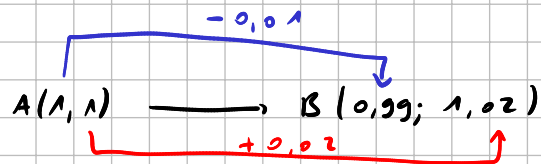
• $\frac{\partial f(A)}{\partial x} = \frac{2}{2} = 1$

• $\frac{\partial f}{\partial y}(A) = \frac{4}{2} = 2$

• $\frac{\partial^2 f}{\partial x^2}(A) = \frac{0}{2^2} = 0$

• $\frac{\partial^2 f}{\partial x \partial y}(A) = \frac{8}{2^2} = 2$

• $\frac{\partial^2 f}{\partial y^2}(A) = \frac{8}{2^2} = 2$



En $B(0.99; 1.02)$ on a $h = -0.01$ et $k = 0.02$

$f(0.99; 1.02) \simeq f(1,1) + 1 \times (-0.01) + 2 \times 0.02$

$+ \frac{1}{2} (0 \times 0.01^2 + 2 \times 2 \times (-0.01) \times 0.02 + 2 \times 0.02^2)$

$\Rightarrow f(B) - f(A) \simeq 0.03 + \frac{1}{2} (-0.0004 + 0.0008) \simeq 0.0302$. $\square$

⊗ **Exercice 16.** Déterminer l'ensemble de définition de chacune des fonctions suivantes puis en déterminer les points critiques et leur nature.

1. $f(x,y) = -y^2 + 4y - x^2 + 4x + 2$;

2. $f(x,y) = x^2 + y^4 - 2y^2$;

3. $f(x,y) = x^2 + e^{-y} + 2y$;

4. $f(x,y) = x^2 - xy + y^2$;

5. $f(x,y) = x^3 + x^2 - xy + y^2 + 4$;

6. $f(x,y) = x^3 + y^3 - 3xy$;

7. $f(x,y) = x^3 + y^3 - 3x - 12y$;

8. $f(x,y) = x^3 + xy^2 - x^2y - y^3$;

9. $f(x,y) = x^4 + y^4 - 2(x - y)^2$;

10. $f(x,y) = y(x^2 + \ln(y^2))$;

11. $f(x,y) = xe^y - 3yx^2$;

12. $f(x,y) = e^x + e^y + e^{-x-y}$;

13. $f(x,y) = (x^2 + y^2)e^{x^2-y^2}$.

Optimisation lib.

pt critique $X_0 \leftarrow$ solution de $\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases}$ (souvent le $\oplus$ dm)

Nature? p -ê donné par le signe de $S = rt - s^2$ en X_0

$r = \frac{\partial^2 f}{\partial x^2}(X_0)$

$s = \frac{\partial^2 f}{\partial x \partial y}(X_0)$

$t = \frac{\partial^2 f}{\partial y^2}(X_0)$

Si $S > 0$: extremum local

$r > 0$: min local

$r < 0$: max local

Si $S < 0$: pt col

Si $S = 0$: pas de CCL

2 possibls

• extremum local

• pt col

1. $f(x,y) = -y^2 + 4y - x^2 + 4x + 2$

f polynôme $D = \mathbb{R}^2$ f est $\mathcal{C}^2$ sur $\mathbb{R}^2$

• $\frac{\partial f}{\partial x} = -2x + 4$

• $\frac{\partial^2 f}{\partial x^2} = -2$

• $\frac{\partial f}{\partial y} = -2y + 4$

• $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 0$

• $\frac{\partial^2 f}{\partial y^2} = -2$

pt critique

solutions de

$$\begin{cases} -2x + 4 = 0 \\ -2y + 4 = 0 \end{cases} \Leftrightarrow \begin{cases} x = 2 \\ y = 2 \end{cases}$$

1 seul pt critique $A(2,2)$.

Nature

$$r = -2 < 0$$

$$s = 0$$

$$t = -2$$

$$\delta = rt - s^2 = 4 > 0$$

A maximum local.

(en fait global car

$\delta > 0$ sur $\mathbb{R}^2 \dots$)

$$2. f(x, y) = x^2 + y^4 - 2y^2$$

f polynôme $\Rightarrow \mathcal{C}^2$ sur $\mathbb{R}^2$

$$\frac{\partial f}{\partial x} = 2x \quad \text{!}$$

$$\frac{\partial^2 f}{\partial x^2} = 2 \quad \text{!}$$

$$\frac{\partial f}{\partial y} = 4y^3 - 4y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 0$$

$$\frac{\partial^2 f}{\partial y^2} = 12y^2 - 4$$

pts critiques
solutions de

$$\begin{cases} \frac{\partial f}{\partial x} = 0 \\ 4y^3 - 4y = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y(y^2 - 1) = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y(y-1)(y+1) = 0 \end{cases}$$

3 pts critiques : $O(0; 0)$, $A(0; 1)$, $B(0; -1)$.

Nature.

	$O(0; 0)$	$A(0; 1)$	$B(0; -1)$
r	2	2	2
s	0	0	0
t	-4	8	8
δ	-8 < 0	16 > 0	16 > 0
Nature	pt col	min local	min local

Par chance, l'exercice commence très tôt ne demande pas de tout relire !

3. $f(x, y) = x^2 + e^{-y} + 2y$

$D = \mathbb{R}^2$ et f est $\mathcal{C}^2$ sur $\mathbb{R}^2$ comme somme de fonctions $\mathcal{C}^2$.

$(e^u)' = u' e^u$

$\cdot \frac{\partial f}{\partial x} = 2x$

$\cdot \frac{\partial^2 f}{\partial x^2} = 2$

$\cdot \frac{\partial f}{\partial y} = -e^{-y} + 2$

$\cdot \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 0$

$\cdot \frac{\partial^2 f}{\partial y^2} = e^{-y}$

pt critique. On résout $\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases} \Leftrightarrow \begin{cases} 2x = 0 \\ -e^{-y} + 2 = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ e^{-y} = 2 \end{cases}$

$\Leftrightarrow \begin{cases} x = 0 \\ -y = \ln(2) \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = -\ln(2) = \ln(\frac{1}{2}) \end{cases}$ Un seul pt critique $A(0; \ln(\frac{1}{2}))$

Nature

$r = 2 > 0$

$s = 0$

$t = e^{\ln 2} = 2$

$\delta = rt - s^2 = 4 > 0$

$\left. \begin{matrix} r = 2 > 0 \\ s = 0 \\ t = 2 \end{matrix} \right\} A \text{ min. local.}$

4. $f(x, y) = x^2 - xy + y^2$ f polynôme donc $\mathcal{C}^2$ sur $\mathbb{R}^2$

$\frac{\partial f}{\partial x} = 2x - y$

$\frac{\partial^2 f}{\partial x^2} = 2$

$\frac{\partial f}{\partial y} = -x + 2y$

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = -1$

$\frac{\partial^2 f}{\partial y^2} = 2$

pt critique $\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases} \Leftrightarrow \begin{cases} 2x - y = 0 \\ -x + 2y = 0 \end{cases} \Leftrightarrow \begin{cases} y = 2x \\ -x + 4x = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$

$O(0; 0)$ seul pt critique.

Nature. $\delta = rt - s^2 = 2 \times 2 - (-1)^2 = 4 - 1 = 3 > 0$; $r = 2 > 0$
 O est un min (global r, s, t évalés constants...)