

L2S3, Visio du 24 octobre 2026, 11h30

* Exercice 15. Soit f définie par $f(x,y) = \ln(1+x^2y^4)$.

Déterminer le développement limité à l'ordre 2 de f au point $A = (1;1)$.

En déduire une valeur approchée de $f(B) - f(A)$ avec $B = (0.99; 1.02)$.

1. $f(x,y) = \ln(1+x^2y^4)$ $\mathcal{D}_f:]0;+\infty[$

or $\forall (x,y) \in \mathbb{R}^2, x^2y^4 \geq 0$ donc $1+x^2y^4 \geq 1 > 0$

donc $1+x^2y^4 \in \mathcal{D}_f$ enfin $\mathcal{D}_f = \mathbb{R}^2$.

f étant la composée d'un polynôme par $\ln$, f est $\mathcal{C}^2$ sur $\mathbb{R}^2$.

Dérivées partielles.

$(\ln u)' = \frac{u'}{u}$

1^{er} ordre

$$\begin{cases} \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} (\ln(1+x^2y^4)) = \frac{2xy^4}{1+x^2y^4} \\ \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (\ln(1+x^2y^4)) = \frac{4x^2y^3}{1+x^2y^4} \end{cases}$$

$(\frac{u}{v})' = \frac{u'v - uv'}{v^2}$

2nd ordre.

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{2xy^4}{1+x^2y^4} \right) = \frac{2y^4(1+x^2y^4) - 2xy^4(2xy^4)}{(1+x^2y^4)^2} \\ &= \frac{2y^4(1+x^2y^4 - 2x^2y^4)}{(1+x^2y^4)^2} = \frac{2y^4(1-x^2y^4)}{(1+x^2y^4)^2} \\ \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{2xy^4}{1+x^2y^4} \right) = \frac{8xy^3(1+x^2y^4) - 2xy^4(4x^2y^3)}{(1+x^2y^4)^2} \\ \text{Schwarz} \quad \frac{\partial^2 f}{\partial x \partial y} &= \frac{8xy^3(1+x^2y^4 - 2x^2y^4)}{(1+x^2y^4)^2} = \frac{8xy^3}{(1+x^2y^4)^2} \\ \frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y} \left(\frac{4x^2y^3}{1+x^2y^4} \right) = \frac{12x^2y^2(1+x^2y^4) - 4x^2y^3(4x^2y^3)}{(1+x^2y^4)^2} \\ &= \frac{4x^2y^2(3(1+x^2y^4) - 4x^2y^4)}{(1+x^2y^4)^2} = \frac{4x^2y^2(3-x^2y^4)}{(1+x^2y^4)^2} \end{aligned}$$

DL₂ $f(A)$ $H(h,k)$

$$f(A+H) = f(A) + \frac{\partial f}{\partial x}(A)h + \frac{\partial f}{\partial y}(A)k + \frac{1}{2} \left[\frac{\partial^2 f}{\partial x^2}(A)h^2 + 2 \frac{\partial^2 f}{\partial x \partial y}(A)hk + \frac{\partial^2 f}{\partial y^2}(A)k^2 \right] + \|H\|^2 \varepsilon(H)$$

$f(B) - f(A) \approx$ car $\lim_{H \rightarrow 0} \varepsilon(H) = 0$

!
 éhová...

$$\text{ex } A(1;1), \quad 1+x^2 y^1 = 1+1=2$$

$$\bullet f(A) = 05EF$$

$$\bullet \frac{\partial f}{\partial x}(A) = 2$$

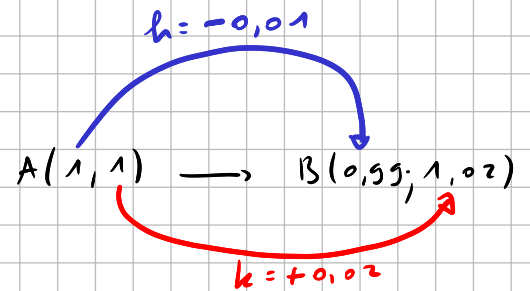
$$\bullet \frac{\partial f}{\partial y}(A) = 4$$

$$\bullet \frac{\partial^2 f}{\partial x^2}(A) = 0$$

$$\bullet \frac{\partial^2 f}{\partial x \partial y}(A) = 8$$

$$\bullet \frac{\partial^2 f}{\partial y^2}(A) = 8$$

Valeu a f fuction ex $B(0,99; 1,02)$



$$f(B) - f(A) = 2 \times (-0,01) + 4 \times (0,02)$$

$$+ \frac{1}{2} \left[0 \times (-0,01)^2 + 2 \times 8 \times (-0,0002) + 8 \times 0,0004 \right]$$

$$= 0,06 + \frac{1}{2} (0 - 0,0032 + 0,0032)$$

$$= 0,06 + \quad \quad \quad = 0,06 \quad (?)$$

à
réviser
à la
calculatrice...

⊛ **Exercice 16.** Déterminer l'ensemble de définition de chacune des fonctions suivantes puis en déterminer les points critiques et leur nature.

1. $f(x,y) = -y^2 + 4y - x^2 + 4x + 2$;
2. $f(x,y) = x^2 + y^4 - 2y^2$;
3. $f(x,y) = x^2 + e^{-y} + 2y$;
4. $f(x,y) = x^2 - xy + y^2$;
5. $f(x,y) = x^3 + x^2 - xy + y^2 + 4$;
6. $f(x,y) = x^3 + y^3 - 3xy$;
7. $f(x,y) = x^3 + y^3 - 3x - 12y$;
8. $f(x,y) = x^3 + xy^2 - x^2y - y^3$;
9. $f(x,y) = x^4 + y^4 - 2(x-y)^2$;
10. $f(x,y) = y(x^2 + \ln(y^2))$;
11. $f(x,y) = xe^y - 3yx^2$;
12. $f(x,y) = e^x + e^y + e^{-x-y}$;
13. $f(x,y) = (x^2 + y^2)e^{x^2-y^2}$.

Optimisation libre ← p-2 du !

$D_f + \mathbb{R}^2$ pts critiques : solutions de $\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases}$

convexité : signe $\delta = rt - s^2$

où $\begin{aligned} r &= \frac{\partial^2 f}{\partial x^2}(A) \\ s &= \frac{\partial^2 f}{\partial x \partial y}(A) \\ t &= \frac{\partial^2 f}{\partial y^2}(A) \end{aligned}$ A pt critique

$\left. \begin{aligned} & \\ & \end{aligned} \right\} \text{extremum local}$

critère : si $\delta > 0 \rightarrow$ extremum local

$\begin{cases} r > 0 : \text{min local} \\ r < 0 : \text{max local} \end{cases}$

si $\delta < 0 \rightarrow$ pt col

si $\delta = 0 \rightarrow$ pas de CCL.

1. $f(x,y) = -y^2 + 4y - x^2 + 4x + 2$

• 1 polynôme défini et $\mathbb{R}^2$ sur $\mathbb{R}^2$

• $\underline{\partial f}$ $\cdot \frac{\partial f}{\partial x} = -2x + 4 \quad \cdot \frac{\partial f}{\partial y} = -2y + 4$

$\cdot \frac{\partial^2 f}{\partial x^2} = -2 \quad \cdot \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 0 \quad \cdot \frac{\partial^2 f}{\partial y^2} = -2$

Sch

• pt critique On résout $\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases} \Leftrightarrow \begin{cases} -2x + 4 = 0 \\ -2y + 4 = 0 \end{cases} (*) \begin{cases} x = 2 \\ y = 2 \end{cases}$

$A(2;2)$ unique pt critique

$r = -2 < 0$, $s = 0$, $t = -2$, $\delta = rt - s^2 = 4 > 0$, CCL: $A(2;2)$ maximum local.

$$2. f(x, y) = x^2 + y^4 - 2y^2$$

f polynôme défini et $\mathcal{C}^2$ sur $\mathbb{R}^2$.

$$\partial P: \bullet \frac{\partial f}{\partial x} = 2x \quad \bullet \frac{\partial f}{\partial y} = 4y^3 - 4y$$

$$\bullet \frac{\partial^2 f}{\partial x^2} = 2 \quad \bullet \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 0 \quad \bullet \frac{\partial^2 f}{\partial y^2} = 12y^2 - 4.$$

pt critiques On résout
$$\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases} \Leftrightarrow \begin{cases} 2x = 0 \\ 4y^3 - 4y = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y(y^2 - 1) = 0 \end{cases}$$

$\Leftrightarrow \begin{cases} x = 0 \\ y(y-1)(y+1) = 0 \end{cases}$

3 pts critiques :
 $O(0;0)$ $A(0;1)$ $B(0;-1)$

	$O(0;0)$	$A(0;1)$	$B(0;-1)$
n	2	2 > 0	2 > 0
s	0	0	0
t	-4	8	8
δ	-8 < 0	16 > 0	16 > 0
CCL	pt col	min. local	min. local

$$(e^u)' = u' e^u$$

$$3. f(x, y) = x^2 + e^{-y} + 2y$$

f est la somme de fonctions définies et $\mathcal{C}^2$ sur $\mathbb{R}^2$

f est définie et $\mathcal{C}^2$ sur $\mathbb{R}^2$.

$$\partial P \quad \cdot \frac{\partial f}{\partial x} = 2x \quad \cdot \frac{\partial f}{\partial y} = -e^{-y} + 2$$

$$\cdot \frac{\partial^2 f}{\partial x^2} = 2 \quad \cdot \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 0 \quad \cdot \frac{\partial^2 f}{\partial y^2} = e^{-y}$$

pts critiques:

$$\text{On résout } \begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases} \Leftrightarrow \begin{cases} 2x = 0 \\ -e^{-y} + 2 = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ e^{-y} = 2 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ -y = \ln 2 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = -\ln 2 \end{cases}$$

$A(0; -\ln 2)$ unique pt critique de f .

Nature $r = 2 > 0, s = 0, t = e^{\ln 2} = 2$

$$s = 4 > 0$$

A min. local de f .



$$4. f(x, y) = x^2 - xy + y^2$$

f polynôme défini et $\mathcal{C}^2$ sur $\mathbb{R}^2$.

$$\cdot \frac{\partial f}{\partial x} = 2x - y \quad \cdot \frac{\partial f}{\partial y} = -x + 2y$$

$$\cdot \frac{\partial^2 f}{\partial x^2} = 2 \quad \cdot \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = -1 \quad \cdot \frac{\partial^2 f}{\partial y^2} = 2$$

$$\text{pt critique: On résout } \begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases} \Leftrightarrow \begin{cases} 2x - y = 0 \\ -x + 2y = 0 \end{cases} \Leftrightarrow \begin{cases} y = 2x \\ -x + 4x = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$$

$O(0;0)$ unique pt critique

Nature? $r = 2 > 0, s = -1, t = 2, s = 4 - 1 = 3 > 0$

O minimum local.

